Estimates and implications of surface eddy diffusivity in the southern ocean derived from tracer transport

John Marshall\textsuperscript{a}, Emily Shuckburgh\textsuperscript{b}, Helen Jones\textsuperscript{a} and Chris Hill\textsuperscript{a}
Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology\textsuperscript{a}, USA
Department of Applied Mathematics and Theoretical Physics, University of Cambridge\textsuperscript{b}, UK
March 20, 2005

Abstract

Effective diffusivities associated with geostrophic eddies in the southern ocean are estimated by numerically monitoring the lengthening of idealized tracer contours as they are strained by surface geostrophic flow observed by satellite altimetry. The resulting surface diffusivities show considerable spatial variability and are large (2000 m$^2$ s$^{-1}$) on the equatorward flank of the Antarctic Circumpolar Current and are small (500 m$^2$ s$^{-1}$) at the jet axis. Regions of high and low effective diffusivity are shown to be collocated with regions of, respectively, weak and strong isentropic potential vorticity gradients. The maps of diffusivity are used, along with climatological estimates of surface wind stress and air-sea buoyancy flux, to estimate surface meridional residual flows and the relative importance of Eulerian and eddy-induced circulation in the streamwised-average dynamics of the Antarctic Circumpolar Current.
1 Introduction

An outstanding problem in large-scale ocean dynamics is the understanding, characterization and representation of tracer transport by geostrophic eddies particularly in highly turbulent regions of the ocean, such as the western boundary currents and the Antarctic Circumpolar Current of the Southern Ocean. Eddy transport in the coarse resolution ocean models used in climate research is parameterized by assuming that eddy tracer flux is related to mean tracer gradient through an eddy diffusivity, $K$. Arguably the key uncertainty in these models is lack of knowledge of the magnitude of the $K$’s and how they vary in space and time. These matters are further complicated because the geographical distribution of eddy transfer is difficult to characterize by in-situ measurements which by their very nature tend to be local.

Hitherto estimates of $K$ have been inferred from observations in a number of different ways:

1. mooring data of timeseries of velocity $v$, and, for example, temperature $T$, can be used to construct $\sqrt{\overline{T^2}}$ and its relationship to $\nabla T$ [see, for example, Bryden and Heath, 1985]. However, the time-series are rather short (a few years at most) making interpretation difficult and very few such estimates have been made, raising doubt about how representative they are. Moreover, and more fundamentally, interpretation of such ‘point’ estimates are seriously compromised by the presence of large non-divergent eddy fluxes which play no role in eddy-mean-flow interaction (Marshall and Shutts, 1981).

2. floats and drifters can be used to estimate lateral diffusivities by measuring components of the diffusivity tensor and dispersion rates, as described in, for example, Davis (1991), Lumpkin et al (2002) and Zhurbas and Oh (2004). However, such diffusivities are not directly related to the mixing properties of, or tracer transport by eddies and not easily interpretable in terms of the eddy diffusivities employed in large-scale ocean models.

3. since the advent of satellite altimetry, our knowledge of eddy statistics at the ocean surface has increased substantially. Holloway (1986) [see also Keffer and Holloway, 1998; Stammer, 1998; Karsten and Marshall, 2002] proposed that eddy diffusivities could be estimated from maps of r.m.s. height field variability using scaling arguments. Kushner and
Held (1998) tested the approach in a meteorological context to infer the divergent part of the eddy temperature flux. They emphasized that estimates based on Holloway’s method are best interpreted as near-surface diffusivities. The approach is compromised, however, by its reliance on scaling arguments and because the diffusivities so derived are only known up to a ‘constant of proportionality’.

In this paper we attempt to arrive at an alternative estimate of ocean eddy diffusivities using a method pioneered by Nakamura (1996) and applied to diagnose tracer transport in the stratosphere by Haynes and Shuckburgh (2000a,b), Allen and Nakamura (2001) and to tracer transport in a laboratory setting by Deese et al (2002). We diagnose transport properties of the eddy field at the ocean’s surface by driving an advection-diffusion equation for idealized tracers using 2-d non-divergent flow derived from altimetric observations and calculating the ‘effective diffusivity’. The attractiveness of the approach is that, as shown by Nakamura (1996), by adopting an area coordinate, the tracer transport problem can be rigorously phrased as a diffusion problem:

\[
\frac{\partial q}{\partial t} = \frac{\partial}{\partial A} \left( \kappa_{\text{eff}}(A) \frac{\partial q}{\partial A} \right)
\]

where \( A \) is the area between a particular \( q \) contour and a reference contour, and \( \kappa_{\text{eff}}(A) \) depends on integrals of \(|\nabla q|^2\) over the area \( A \) (the mathematical framework is set out in an appendix — see Section 7.2). The effective diffusivity diagnostic is based on identifying the enhancement of diffusion that arises through the effects of eddies stretching and folding tracer contours. In mixing regions tracers are vigorously stretched into complex geometrical shapes with tight gradients, and this leads to large values of effective diffusivity. Tracer geometry in regions of barriers is usually smooth, creating localised small values of effectively diffusivity — so small as to keep the flux minimal despite the often large tracer gradients. Because \( \kappa_{\text{eff}} \) is diagnosed directly from tracer fields being dispersed by eddies, it is more obviously connected to the diffusivities employed in large-scale ocean models than, for example, the diffusivity deduced from the dispersal of floats.

In this study we use Nakamura’s approach to yield surface eddy diffusivities associated with horizontal geostrophic eddies measured by altimetry. Surface diffusivities are of special interest and importance because, firstly (and as reviewed by Treguier et al, 1997), quasi-geostrophic theory tells us
that eddy-mean flow interaction involves interior potential vorticity fluxes and horizontal buoyancy fluxes at the boundary. The two are related to one another, but our diagnostic approach directly yields the surface diffusivities. Secondly, horizontal boundary eddy fluxes appear as a forcing term on the rhs of the residual-mean buoyancy equation (see Plumb and Ferrari, 2005) and can be used, as in Karsten and Marshall (2002), to infer the eddy-induced horizontal circulation at the surface of the ocean associated with eddies.

Our paper is set out as follows. In Section 2 we describe how we use altimeteric data to drive an advection diffusion problem for idealized tracers over the southern ocean. In Section 3 we present our estimates of eddy diffusivities and discuss the robustness of our results. We find that the effective diffusivity exhibits interesting spatial variation, having elevated values (1500 to 2000 m² s⁻¹) on the equatorward flank of the Antarctic Circumpolar Current (ACC) which diminish as the core of the current is approached, to increase again on its poleward flank. We interpret this distribution in terms of the geographical distribution of strain-rate and the large-scale potential vorticity (PV) distribution, the core of the ACC being characterized by large PV gradients which inhibit lateral dispersion. In Section 4 we make use of the effective diffusivity to quantify the role of eddies in the streamwise-average residual-mean dynamics of the ACC, comparing wind and eddy-induced cross-stream transport. We argue that mixing by eddies on the equatorial flank of the ACC induces a cross-stream flow directed poleward in opposition to the equatorward flow driven directly by the wind. The convergence of these opposed currents induces subduction from the surface into the interior which we identify as Antarctic Intermediate Water. We conclude in Section 5.

2 Diagnostic approach

2.1 Theoretical background

We are concerned with the problem of a passive tracer, q, advected by a two-dimensional non-divergent velocity field v, described by the evolution equation:

\[ \frac{\partial q}{\partial t} + v \cdot \nabla q = k \nabla^2 q \]  

(2)

where k is a constant diffusivity. The advection diffusion problem has been the subject of numerous investigations dating back to at least Batchelor
(1959) — for recent reviews see Majda and Kramer (1999) and Pierrehumbert (2000). In the limit of large Peclet number — \( Pe = \frac{V L}{k} \) where \( V \) and \( L \) are the characteristic scales of velocity and eddy length, respectively — diffusion is so small as to make the effect of diffusion appreciable only on very small scales. In the presence of the stirring action of the velocity field, \( q \) contours are stretched in to complex geometrical shapes driving a general increase in the gradients of \( q \). In the absence of diffusion, the area of fluid demarcated by two tracer contours cannot change, irrespective of how convoluted those contours may become. However, in the presence of a \( k \), enhanced gradients of \( q \) leads to enhanced mixing and the area contained between \( q \) contours changes. By carrying out a census of \( |\nabla q|^2 \) over the area contained within \( q \) contours, an effective eddy diffusivity can be computed as follows.

Nakamura (1996) (see also Winters and D’Asaro, 1996) shows that in area coordinates Eq.(2) reduces to Eq.(1) with the diffusivity defined by:

\[ \kappa_{eff} = kL_{eq}^2 \]  

where \( L_{eq} \) is the ‘equivalent length’ of a \( q \) contour (see appendix, Eq.18). Note that \( \kappa_{eff} \) has units of \( m^4 s^{-1} \) because — see Eq.(1) — it represents diffusion of \( q \) in area coordinates. To obtain a diffusivity in physical space with units of \( m^2 s^{-1} \), we define a quantity

\[ K_{eff} = k\frac{L_{eq}^2}{L_{min}^2} \]  

where \( L_{min} \) is the minimum (unstrained) length of a \( q \) contour.

As discussed in more detail in Section 3.1, for \( Pe = \frac{V L}{k} \ll 1 \), one would expect diffusion to dominate over advection and hence \( K_{eff} \) to tend to \( k \), with \( L_{eq} \) tending to \( L_{min} \). However, if \( Pe \gg 1 \), then one might expect \( K_{eff} \) to be independent of the magnitude of \( k \), because it is the stirring of tracers by the large scale velocity field that controls the gradients of \( q \) on which the microscale diffusion acts. This has been shown to be true for the case of simple chaotic advection flows (Shuckburgh and Haynes, 2003) and we will study whether it is also true for the oceanographic flows under consideration here.

\[ \text{For the interest of the oceanographic community, we append a derivation of how one goes from Eq.(2) to (1) which parallels Walin’s (1982) formulation of water mass transformation, as reviewed in Marshall et al (1999). The theoretical framework of Nakamura (1996) and Walin (1982) are closely related to one another, although they were addressing rather different problems.} \]
The action of diffusion is enhanced through differential advection which, by the stretching and folding of material lines, produces small scale structures in the tracer field. This is manifested by the enhancement of equivalent length $L_{eq}$ over the minimum length $L_{min}$, resulting in an effective diffusivity which is larger than $k$ — see Eq. (4).

2.2 Solving the advection-diffusion tracer equation driven by altimetry

To apply the method described above we chose to focus on the southern ocean polewards of $30^\circ S$. Knowledge of surface diffusivities in the southern ocean is of great importance to understanding its dynamics and the role of eddy fluxes therein — see for example Rintoul et al (2001), Karsten and Marshall (2002), Marshall and Radko (2003), Olbers et al (2004), and Section 4.1.

Altimetric observations over the southern ocean — every 10 days on a $1/4^\circ$ lat-lon grid — were used to provide the driving velocity field. Details of the altimetric data and the geoid model used in our study are given in the appendix. Before doing any serious calculations we began by inspecting movies of altimetric height. It was reassuring to observe continuity of eddy features over time in the gridded data. For flows in a chaotic advection regime, it is the coarse-grained large-scale velocity field which is of greatest import to tracer evolution and hence to calculation of effective diffusivity. Stratospheric flows appear to fall in this regime (e.g., Haynes and Shuckburgh 2000a,b), and there is some evidence that this is also true for surface-ocean flows (Waugh, 2005, personal communication). This encouraged us to move on to directly implement our advection diffusion problem driven by the evolving, coarse grained altimetric fields. First the data were interpolated on to the higher resolution grid on which the advection diffusion calculation was carried out. The grids used — at $1/20^\circ$ and $1/100^\circ$ — are set out in Table 1. The interpolated altimetric-derived velocity field were then made non-divergent as is now described.

2.2.1 Preparation of a non-divergent velocity field

If $\eta$ is the altimetric height of the sea surface relative to the geoid, then the geostrophic relation yields the geostrophic current:
\[ \mathbf{v}_g = \frac{g}{f} \hat{\mathbf{z}} \times \nabla \eta \]  \hspace{1cm} (5)

where \( g \) is the acceleration due to gravity, \( f = 2\Omega \sin \text{lat} \) is the Coriolis parameter and \( \hat{\mathbf{z}} \) is a unit vector in the vertical. Because of (i) variations in \( f \) and (ii) the presence of boundaries where the total normal velocity vanishes \( (\mathbf{v} \cdot \mathbf{n} = 0 \), where \( \mathbf{n} \) is a unit vector normal to the boundary), Eq.(5) will yield a velocity field that is divergent. We therefore set:

\[ \mathbf{v} = \mathbf{v}_g + \nabla \chi \]  \hspace{1cm} (6)

where \( \nabla \chi \) is a (divergent) ‘adjustment’ to the altimetric velocity which renders the sum, \( \mathbf{v} \), non-divergent: \( \nabla \cdot \mathbf{v} = 0 \). Thus, taking the horizontal divergence of Eq.(6), we obtain:

\[ \nabla^2 \chi = -\nabla \cdot \mathbf{v}_g \]  \hspace{1cm} (7)

with boundary condition:

\[ \nabla \chi \cdot \mathbf{n} = -\mathbf{v}_g \cdot \mathbf{n} \]  \hspace{1cm} (8)

Thus the elliptic equation (7) was inverted for \( \chi \) over the southern ocean with boundary conditions given by Eq.(8) and the non-divergent velocity field then constructed from Eq.(6).

The above calculations were carried out on the numerical grid on which the advection-diffusion problem was solved.

### 2.2.2 Time-stepping the advection diffusion problem

Using a time series of non-divergent flow fields, \( \mathbf{v} \), generated according to the procedure laid out above, we advect a passive tracer \( q \). The altimeter data yields one set of flow fields for each ten-day period. We calculate an annually periodic set of thirty six flow fields and evolve \( q \) using the annual cycle according to

\[ \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q = k \nabla^2 q. \]  \hspace{1cm} (9)

An initial distribution for \( q \) is chosen to align with the time-mean geostrophic stream lines as shown in Fig.1(a). This is to ensure that as the tracer evolves
there is sufficient contrast in tracer values between the different transport regions to allow a tracer-based coordinate system to be useful.

Eq.(9) is stepped forward numerically on the sphere using the infrastructure of the MITgcm (Marshall et al, 1997). We choose to use an Adams-Bashforth timestepping scheme in conjunction with a simple centered second order discretization in space which conserves $q$ and $q^2$ and introduces no spurious diffusion. We prefer not to use higher-order or ‘limited’ schemes that, for example, conserve extrema, because they introduce diffusion that would compete with the explicit process, $k\nabla^2 q$, that is central to the Nakamura algorithm — see Eq.(3).

Fig.1 shows a) the initial tracer distribution and b) the tracer after one year of integration from a simulation at a resolution of $1^{\circ}$ resolution with a diffusivity of $k = 50 \text{m}^2\text{s}^{-1}$ and c) a zoom in on a section of the flow. A large number of such calculations were carried out, as set out in Table 1. Many are of subdomains — patches embedded in the larger-scale flow — and some at a resolution as high as $\frac{1}{100^{\circ}}$, much higher resolution than was possible in the global domain. This enabled the explicit diffusivity to be reduced to low levels, allowing our calculations to span over a larger range of $Pe$. Fig. 2 shows a number of patch calculations carried at different resolutions and values of $k$ and hence $Pe$.

Because the velocity field is very smooth at small scales, a good description of the evolution of the $q$ field is $q_t + S_x q_x = q_{xx}$ where $S = \frac{V}{L}$ is the strain rate, where $V$ and $L$ are typical scales for eddy speed and size. A balance between advection and diffusion occurs on the scale $\delta = \sqrt{\frac{L}{S}}$ — a ‘Batchelor scale’. The $Pe$ number for these flows — comparing the advective timescale $\frac{L}{V}$ with the diffusive timescale $\frac{k^2}{S}$ — is then $Pe = \frac{V L}{S k} = \frac{S L^2}{S k} = \left(\frac{L}{\delta}\right)^2$. From Table 1, we see that in our numerical experiments the Batchelor scale exceeds the gridspacing for all but the very smallest values of $k$ used at the various resolutions. The results from these experiments must therefore be considered suspect.

The tracer field is extracted from the advection-diffusion simulation at regular time intervals and the effective diffusivity calculated from it as described in the following section.
Figure 1: (a) Time-mean geostrophic streamfunction, $\Psi_g$, (dotted contours are negative: c.i. $2 \times 10^6 \text{m}^2 \text{s}^{-1}$) and initial tracer distribution (colors) (b) Instantaneous tracer distribution, ranging in value from 0 to 1, after 1 year of integration at $1/20^\circ$ with $k = 50 \text{m}^2 \text{s}^{-1}$ (c) a zoom in on a section of the global flow.
Figure 2: Tracer distributions after 1 year of integration carried out on a patch at various resolutions and diffusivities. The full suite of experiments is set out in Table 1.
<table>
<thead>
<tr>
<th>Expt</th>
<th>Domain</th>
<th>Res (° lat)</th>
<th>$k$ (m²·s⁻¹)</th>
<th>$K_{eff}$ (m²·s⁻¹)</th>
<th>$P_e$</th>
<th>$\delta$ (km)</th>
<th>$N_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>global</td>
<td>$\frac{1}{20}$ (≡ 5 km)</td>
<td>10</td>
<td>840</td>
<td>258</td>
<td>3.4</td>
<td>84</td>
</tr>
<tr>
<td>2</td>
<td>[43-53S]</td>
<td>''</td>
<td>25</td>
<td>1125</td>
<td>102</td>
<td>5.4</td>
<td>45</td>
</tr>
<tr>
<td>3</td>
<td>red dots</td>
<td>''</td>
<td>50</td>
<td>1280</td>
<td>52</td>
<td>7.7</td>
<td>26</td>
</tr>
<tr>
<td>4</td>
<td>''</td>
<td>''</td>
<td>100</td>
<td>1700</td>
<td>26</td>
<td>10.9</td>
<td>17</td>
</tr>
<tr>
<td>5</td>
<td>patch I at $\frac{1}{20}$</td>
<td>$\frac{1}{20}$ (≡ 5 km)</td>
<td>10</td>
<td>280</td>
<td>168</td>
<td>4.3</td>
<td>28</td>
</tr>
<tr>
<td>6</td>
<td>[43-53S],[43-53E]</td>
<td>''</td>
<td>100</td>
<td>630</td>
<td>17</td>
<td>13.5</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>patch I at $\frac{1}{100}$</td>
<td>$\frac{1}{100}$ (≡ 1 km)</td>
<td>1</td>
<td>560</td>
<td>1763</td>
<td>1.3</td>
<td>557</td>
</tr>
<tr>
<td>8</td>
<td>[43-53S],[43-53E]</td>
<td>''</td>
<td>10</td>
<td>610</td>
<td>176</td>
<td>4.2</td>
<td>61</td>
</tr>
<tr>
<td>9</td>
<td>dark blue dots</td>
<td>''</td>
<td>100</td>
<td>750</td>
<td>18</td>
<td>13.3</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>patch II</td>
<td>$\frac{1}{20}$ (≡ 5 km)</td>
<td>10</td>
<td>720</td>
<td>354</td>
<td>2.9</td>
<td>72</td>
</tr>
<tr>
<td>11</td>
<td>[30-59S],[40-100E]</td>
<td>''</td>
<td>50</td>
<td>1140</td>
<td>71</td>
<td>6.6</td>
<td>23</td>
</tr>
<tr>
<td>12</td>
<td>patch III</td>
<td>$\frac{1}{20}$ (≡ 5 km)</td>
<td>10</td>
<td>490</td>
<td>185</td>
<td>4.1</td>
<td>49</td>
</tr>
<tr>
<td>13</td>
<td>[38-63S],[105-95W]</td>
<td>''</td>
<td>25</td>
<td>690</td>
<td>74</td>
<td>6.5</td>
<td>28</td>
</tr>
<tr>
<td>14</td>
<td>light turquoise dots</td>
<td>''</td>
<td>50</td>
<td>920</td>
<td>37</td>
<td>9.1</td>
<td>18</td>
</tr>
<tr>
<td>15</td>
<td>''</td>
<td>''</td>
<td>100</td>
<td>1260</td>
<td>19</td>
<td>12.9</td>
<td>13</td>
</tr>
<tr>
<td>16</td>
<td>''</td>
<td>''</td>
<td>500</td>
<td>2630</td>
<td>3.7</td>
<td>28.9</td>
<td>5.3</td>
</tr>
<tr>
<td>17</td>
<td>global</td>
<td>$\frac{1}{4}$ (≡ 25 km)</td>
<td>100</td>
<td>952</td>
<td>25</td>
<td>11.0</td>
<td>9.5</td>
</tr>
</tbody>
</table>

Table 1: Tabulated numerical values from our various tracer advection/diffusion calculations. Values of diagnosed $K_{eff}$ are averages over the whole computational domain and so integrate out all spatial gradients: $P_e = \frac{V L}{k}; N_u = \frac{K_{eff}}{k}$ and $\delta = \sqrt{\frac{k}{S}}$ where $S = \frac{V}{L}$. Data from this table is plotted in the ($N_u, P_e$) plot in Fig. 3: the color coding used in the plot is indicated in the table.
3 Estimates of surface eddy diffusivity

To compute $K_{eff}$, the gradient of the tracer is calculated at each grid point, and its square integrated over the area bounded by the desired tracer contour. This integrated $|\nabla q|^2$ is then differentiated with respect to area by taking finite differences. The resulting quantity is then divided by the square of the areal gradient of the tracer, to obtain $L_{eq}^2(q)$ as defined in Eq.(18).

To obtain $L_{min}^2$ we take advantage of the fact that $L_{eq}$ tends to $L_{min}$ for $Pe << 1$. We therefore conduct an advection-diffusion integration with a large diffusivity $k$ — chosen, after sensitivity studies, to be $k = 10^4$ m$^2$s$^{-1}$ — from which we estimate $L_{min}$. The effective diffusivity, $K_{eff}$, is then computed from Eq.(4).

There is an initial period of adjustment, lasting a few eddy turnover times, during which the initial tracer field adjusts to align with the flow. Following this adjustment time, the effective diffusivity remains approximately constant on each $q$ contour, varying only to reflect changes in time of the velocity field. We will present the results of calculations representative of this post-adjustment phase taken at one year of integration, the time chosen to show the tracer distributions in Figs.1 and 2.

3.1 Magnitude of $K_{eff}$

In order to study the dependence of our results on non-dimensional parameters, in Fig.3 we plotted them all up in $(N_u, Pe)$ space where $N_u = \frac{K_{eff}}{k}$ is a Nusselt number and $Pe = \frac{S L^2}{k}$ is the Peclet number and $S \sim \frac{V}{L}$ is calculated from a finite-time Lyapunov exponent calculation$^2$. What might we expect the $(N_u, Pe)$ plot to look like?

As discussed above, tracer stirring will create gradients on the Batchelor scale $\delta = \sqrt{\frac{k}{S}}$. The diffusive flux across filaments of scale $\delta$ will then be $F = \frac{k \Delta q}{\delta}$ where $\Delta q$ is the jump in tracer value across the filament. But since $\Delta q = GL$ where $G$ is the large-scale gradient and $L$ is the eddy scale, then $F = \frac{k}{\delta} GL$.

There are two limit cases: when $Pe$ is large and when it is small.

$^2$We calculate finite-time Lyapunov exponents by following the separation of Lagrangian trajectories with initial perturbations of $10^{-6}$ in latitude and longitude and using the so-called ‘pull-back’ method to remove non-linear effects due to amplification (see Shuckburgh and Haynes 2003)
Figure 3: Nusselt number based on the effective diffusivity \( N_u = \frac{K_{\text{eff}}}{k} \) against a Peclet number based on the stretching rate \( P_e = \frac{V_L}{k} \). The continuous line has a slope of unity: the dotted line a slope of one half.
1. If $Pe$ is large, the increase in the tracer gradient will be accompanied by an increase in the length of the tracer contours such that $L_{\text{contour}} \times \delta \sim L^2$ (ensuring that the area within a tracer contour is roughly conserved). Thus the flux across the tracer contour will scale like $FL_{\text{contour}} = \frac{k}{3} L^3 G = SL^3 G = VL \Delta q$. We thus identify a $K_{\text{eff}} \sim VL$ i.e. independent of $k$.

2. If $Pe$ is small, then $L_{\text{contour}} \sim L$ and so $FL_{\text{contour}} = \frac{k}{3} L^2 G = \frac{k}{3} L \Delta q = \sqrt{kVL}\Delta q$ and $K_{\text{eff}}$ scales like $\sqrt{k}$.

The above scaling suggests that a line of slope unity in $(N_u, Pe)$ space implies that $K_{\text{eff}}$ is dependent only on the rate of straining of tracer gradients by the large-scale flow and independent of small-scale mixing processes. A line of slope 0.5 suggests that the largescale diffusive flux is set by processes right down on the Batchelor scale, $\delta$ i.e the rate controlling process is diffusion across the filaments whose further scale collapse is inhibited by small scale mixing, $k$. The $(N_u, Pe)$ plot in Fig.3 includes, in black dots, the results for $Nu$ and $Pe$ taken from many chaotic advection calculations using a simple analytic flow (see Shuckburgh and Haynes, 2003). These chaotic advection calculations congregate around a line with a slope that is just shy of unity, suggesting that $K_{\text{eff}}$ for these flows is independent of $k$. The results of the altimetric-driven runs — set out in Table 1 — are shown by the colored dots using quantities averaged over the domain of interest (the color code is given in the table). We see that these points all fall on lines with slopes that lie between 1 and 0.5. The best fit line has a slope of 0.76 suggesting that $K_{\text{eff}} = k \left( Pe \right)^{0.76}$ and implying a $K_{\text{eff}}$ that depends on $k^{0.24}$.

Fig.4 shows $K_{\text{eff}}$ plotted as a function of ‘equivalent latitude’ for $k = 10 \text{ m}^2 \text{s}^{-1}$ and $k = 50 \text{ m}^2 \text{s}^{-1}$. We see a very similar spatial variation of $K_{\text{eff}}$ (discussed below) but values of $K_{\text{eff}}$ are larger (by a factor of $\sim 5^{0.24} = 1.47$) for $k = 50 \text{ m}^2 \text{s}^{-1}$ than for $k = 10 \text{ m}^2 \text{s}^{-1}$. Note that as $Pe$ becomes smaller, one might expect $Nu$ to approach unity, as indeed it does and must (because when explicit diffusion dominates $K_{\text{eff}} \rightarrow k$). As $Pe$ becomes smaller, the roll-off to a gently sloping line begins in the range $10 \lesssim Pe \lesssim 50$. From Table 1 we see that $Pe \approx 50$ when $k \approx 50 \text{ m}^2 \text{s}^{-1}$. Thus at these rather high values of $k$ we may indeed expect to observe some dependence of $K_{\text{eff}}$ on $k$.

How large might $k$ be expected to be in the mixed layer? Recent work (Raffael Ferrari and collaborators, personal communication, 2005) — see also Haine and Marshall (1998) — suggests that buoyancy gradients enhanced by
stirring through interior baroclinic instability, are prone to non-hydrostatic instability of the mixed layer which introduce a lateral mixing in the range of 10 to 100 m² s⁻¹. With these values of $k$, the $Pe$ number is not particularly large (perhaps in the range 10 to 250), suggesting the mixed layer is in a regime in which $K_{\text{eff}}$ is perhaps not independent of $k$. Indeed it is in just this range of $Pe$ number that points in $(N_u, Pe)$ space tend to role off the gradient 1 line in Fig.3. Another possibility is that the weak dependence of $K_{\text{eff}}$ on $k$ found here is of numerical origin due to the use of second-order finite differences which generate noise on the grid-scale, false extrema, small scales and numerical dispersion.

In the remainder of this paper we choose to analyze in detail the results of our calculations at a resolution of $1/20^\circ$ with a $k = 50$ m² s⁻¹, indicated by the large red dot in Fig.3.

### 3.2 Spatial variation of $K_{\text{eff}}$

From Fig.4a) it can be seen that $K_{\text{eff}}$ takes a minimum value of around 750 m² s⁻¹ at about $\phi_e = 55^\circ$, with higher values on either side, reaching a maximum of about 2000 m² s⁻¹ at about $\phi_e = 35^\circ$.

The values of $K_{\text{eff}}$ associated with each tracer contour $q$ are plotted in Fig.4b), with the mean streamlines overlaid. We have chosen not to plot the values of $K_{\text{eff}}$ for tracer contours closest to the northern and southern boundaries since the calculation method can result in these values being spurious. It is important to note that $K_{\text{eff}} (\phi_e)$ represents an average value along a tracer contour $q$, and so the two-dimensional plot contains essentially one-dimensional information: the two-dimensional structure comes solely from the tracer distribution $q$. Variations in the effective diffusivity along a tracer contour are not represented. For the purpose of this study we will be interested in the stream-wise average properties of the effective diffusivity for which the along-tracer variation will be of lesser importance. For studies of a particular geographical region, the effective diffusivity calculation should be repeated on a patch representing that geographical region alone, as in Fig.2.

From Fig.4b) it can be seen that the low values of eddy diffusivity ($K_{\text{eff}} < 1000$ m² s⁻¹, shown in dark blue) are closely associated with the ACC. Poleward of this jet there are slightly higher values of eddy diffusivity, and equatorward the values reach more than 2000 m² s⁻¹ (orange/red). The magnitudes and geographical distribution of the diffusivities are consistent with those obtained using an adjoint model [(Fig.12, Ferreira, Marshall and He-]}
Figure 4: Results of a calculation at 20° resolution (a) $K_{eff}$ plotted as a function of equivalent latitude for the case with $k = 50 \text{ m}^2\text{s}^{-1}$ indicated by the large red dot in Fig. 3 and for $k = 10 \text{ m}^2\text{s}^{-1}$ (b) $K_{eff}(q)$ for $k = 50 \text{ m}^2\text{s}^{-1}$ plotted as a 2-d map where $q$ is the tracer distribution after 1 year. The mean geostrophic streamline is superimposed.
imbach, 2004)). The pattern of weak eddy diffusivities associated with a vortex-edge or jet, and stronger eddy diffusivities outside the vortex is seen repeatedly in geophysical flows, plasma physics and simple analytic flows, as discussed in Shuckburgh and Haynes, 2003. Perhaps the most prominent example is the stratospheric polar vortex, which is characterized by a strong barrier at its edge and a “surf-zone” of strong eddy mixing equatorward of this. The barrier effect is thought to be a consequence of the quasi-elastic resilience engendered by the strong potential vorticity gradients at the vortex edge — see Juckes and McIntyre (1987). In support of this conjecture, in Fig.5 we plot the observed isentropic potential vorticity distribution \( f \frac{\partial \sigma}{\partial z} \) on the \( \sigma = 36.6 \) isopycnal surface (details of the computation are given in Section 4 of Marshall et al, 1993). A striking feature is the collar of high PV due to very stable stratification around the pole, a sharp PV gradient roughly coincident with the core of the ACC, and much smaller values of PV in a broad region on the equatorial flank. Indeed the PV field here on the equatorial flank is remarkably uniform. The regions of high diffusivity coincide with those of weak PV gradients: regions of low diffusivity with high PV gradients — compare Fig.4 with Fig.5.

4 Implications of new estimates of K for southern ocean dynamics

4.1 The Deacon Cell of the southern ocean

What inferences can we draw from our \( K_{eff} \) distributions about the role of eddies in the dynamics of the Antarctic Circumpolar Current? Because of the absence of zonal pressure gradients in the ACC, eddies must play a central role in momentum and buoyancy budgets — see, for example, the review by Rintoul et al (2001) and the theoretical model of Marshall and Radko (2003).

Taking a streamwise average of the along-stream residual momentum equation in the planetary geostrophic limit (see Eq.(20) and the discussion in the appendix) the pressure gradient term vanishes on those contours that circumnavigate the globe and we obtain:

\[
-f \tau_{res} = \frac{1}{\rho_{ref}} \frac{\partial (\tau_x^u + \tau_x^e)}{\partial z}
\]

where \( \tau_{res} \) is understood to be the residual current normal to the mean surface.
Figure 5: The isentropic potential vorticity \( f \frac{\partial \sigma}{\partial z} \) distribution on the \( \sigma = 36.6 \) isopycnal surface (potential density referenced to 2 km). Details of the computation are given in Marshall et al (1993).
streamlines and the overbars represent a temporal and streamwise average. Integrating down from the surface, where $\tau^w = \tau_{\text{wind}}$, is the zonal surface wind stress and $\tau^e = 0$, to the bottom of the diabatic layer at depth $h_s$, where $\tau^w = 0$ and $\tau^e$ is given by Eqs.(24) and (26), we find, using $v_{\text{res}} = -\frac{\partial \psi_{\text{res}}}{\partial z}$ with $\psi_{\text{res}} = 0$ at $z = 0$,

$$\psi_{\text{res}}_{z=-h_s} = \frac{-\tau_{\text{wind}}}{\rho_{\text{ref}} f} \underbrace{\psi_{\text{Eu}}}_{z=-h_s} + \left(\underbrace{\frac{\psi^b}{b_z}}_{\psi^*}\right)_{z=-h_s} \tag{11}$$

We see that $\psi_{\text{res}}$ at the bottom of the diabatic layer is made up of an Eulerian-mean contribution due to the Ekman layer ($\psi_{\text{Eu}}$) and an eddy-induced contribution due to lateral eddy buoyancy fluxes ($\psi^*$).

In the nomenclature of Speer et al. (2000), we can call $\psi_{\text{Eu}}$ the Deacon Cell (after Deacon, 1937) and $\psi_{\text{res}}$, the diabatic Deacon Cell. The Deacon Cell can be inferred from the wind-stress shown in Fig.6. The zonal wind-stress is strong and persistent around the southern ocean reaching, when averaged along mean streamlines, a maximum of some 0.14 N m$^{-2}$ just on the equatorward flank of the ACC. Three different climatologies are presented to give an impression of the uncertainty in these estimates. Although there are considerable differences, a broad pattern emerges. Fig.7 shows $\psi_{\text{Eu}}$ deduced by simply averaging the three data sets (assuming that none is superior to another), integrating along mean streamlines and dividing by the Coriolis parameter, $f$. We see that $\psi_{\text{Eu}}$ reaches a magnitude of some 30 Sv at 50°S, with surface flow directed away from Antarctica.

We can estimate the residual flow as follows. As shown in Marshall (1997) and applied in Speer et al. (2000) and Marshall and Radko (2003), $\psi_{\text{res}}_{z=-h_s}$ can be inferred from the buoyancy budget of the mixed layer thus:

$$\psi_{\text{res}}_{z=-h_s} = \frac{\widetilde{B}}{b_y} \tag{12}$$

where $\widetilde{B}$ is the net buoyancy forcing of the mixed layer [including the contribution of diabatic eddy fluxes — see Eq.(12) of Marshall and Radko (2003) and the definition of $F_{\text{res}}$ in the appendix] and $b_y$ is the meridional buoyancy gradient at the base of the mixed layer. Fig.8 shows three estimates of the air-sea buoyancy flux. The difference between them is even larger than for the surface wind-stress, not least due to uncertainties in estimates of evap-
Figure 6: (top) The mean zonal wind-stress for the period 1980-2000 from the NCEP reanalysis (in $Nm^{-2}$). (bottom) The streamwise average wind-stress computed from (i) NCEP (ii) Southampton and (iii) the Da Silva reanalysis products plotted against the mean geostrophic streamfunction ($\Psi_g$). The mean latitude of the mean streamfunction is also indicated.
Figure 7: Estimates of the streamwise-averaged values of (i) $\psi^{Eu}$ from Eq.(11) using the three analyzed fields of surface stress shown in Fig.6: error bars represent ± one standard deviation assuming that none of the three wind stress products is superior to any other (ii) $\psi_{res}$ deduced using Eq.(12) from analyzed fields of surface buoyancy fluxes and a WOCE climatology to obtain $\overline{\sigma}$ (iii) the difference $\psi_{res} - \psi^{Eu}$ (iv) $\psi^*$ from Eq.(13) using our estimates of $K_{eff}$ and observation of isopycnal slope at the base of the diabatic layer, $h_s$. The grey region gives the range of possible $\psi^*$ based on the two $K_{eff}$ estimates shown in Fig.4(top), corresponding to $k = 10$ and $k = 50$ m$^2$s$^{-1}$.
oration and precipitation \((E - P)\). Nevertheless again a consistent broad pattern emerges. On the axis of and spreading poleward of the ACC, there is buoyancy flux in to the ocean — just equatorward of the ACC, the flux is much smaller in magnitude and, if anything, directed out of the ocean. The geographical distribution of the broad patterns of air-sea heat flux (the largest contributor to the air-sea buoyancy flux) can be seen in the top panel of Fig.8 — the yellow/orange regions indicate heat flux in to the ocean, and the blue regions heat flux out of the ocean. The streamwise average \(\psi_{\text{res}}\) computed from Eq.(12) (evaluated using \(\overline{b}_y\) at the base of the mixed layer computed from a gridded WOCE climatology due to Gouretski and Jancke, 1998) is one of the curves shown in Fig.7. Note that here we have set \(\overline{B} = B_s\), the net air-sea buoyancy flux and neglected lateral eddy \((F_{\text{res}})\) contributions to \(\overline{B}\). We see that our estimate of \(\psi_{\text{res}}\) is markedly different from \(\psi^{Eu}\), being considerably smaller in magnitude and exhibiting much more meridional structure. The \(\psi_{\text{res}}\) pattern suggests upwelling of fluid polewards of \(56^\circ\)N of some \(20\) Sv and downwelling of the same magnitude between \(56^\circ\)N and \(50^\circ\)N, not inconsistent with other estimates (see, for example, Karsten and Marshall, 2002).

Also plotted in Fig.7 is the difference \(\psi_{\text{res}} - \psi^{Eu}\) which — see Eq.(11) — can be identified with the eddy-induced streamfunction \(\Psi^*\). If eddies are to substantially balance \(\psi^{Eu}\) they must achieve a poleward volume transport of some \(30\) Sv. We can use our estimate of \(K_{\text{eff}}\) to see whether this is possible or likely.

In Fig.7 we therefore also plot the streamwise average of the \(x\)—component of Eq.(26):

\[
\Psi^* = \frac{\psi^{Eu}}{b_z} = K_{\text{eff}} s_{\rho}
\]

where \(s_{\rho} = -\frac{\overline{b}_y}{b_z}\) is the isopycnal slope at the base of the mixed layer as computed from the WOCE climatology of Gouretski and Jancke (1998). It is very encouraging to observe that \(\Psi^*\) tracks \(\psi_{\text{res}} - \psi^{Eu}\) rather closely, lending strong support to the idea that transient eddies are playing an order one role in balances in the southern ocean.

\(^3\)Due to the presence of meridional boundaries which support zonal pressure gradients, setting \(\Psi^* = \psi_{\text{res}} - \psi^{Eu}\) is not strictly valid, except for the flow that passes through Drake passage.
Figure 8: (top) The mean net air-sea heat flux for the period 1980-2000 from the NCEP reanalysis (in N m$^{-2}$) Blue colors indicate regions where the heat flux is out of the ocean, yellow-orange where it is directed in to the ocean. (bottom) The streamwise average mean net air-sea heat flux computed from (i) NCEP (ii) Southampton and (iii) the Da Silva reanalysis products plotted against the mean geostrophic streamfunction ($\Psi_g$). The mean latitude of the mean streamfunction is also indicated.
Figure 9: The zonal component of eddy stress (in N m$^{-2}$) computed from $\tau_x^e = \rho_{ref} f K_{eff} \bar{s}_\rho$ using our estimate of $K_{eff}$, Fig. 4a), using the $k = 50$ curve.
4.2 Maps of Eddy stresses in the southern ocean

An appealing way of assessing the importance of transient eddy processes in southern ocean dynamics is to again adopt a ‘residual-mean’ perspective and use our effective diffusivity to compute the zonal eddy stress — given by (see Eqs.(24) and (26)) \( \tau_x^e = \rho_{ref} f K_{eff} s_\rho \) — and compare it to the surface wind-stress.

The zonal stress so computed is plotted in Fig.9 in units of N m\(^{-2}\). We see that over the axis of the ACC it is comparable in strength to that of the surface wind stress (compare with Fig.6). One should remember the asymmetry between \( \tau_x^e \) and \( \tau_x^w \). The absolute value of the eddy stress initially always increases in magnitude with depth from zero at surface. Given that typically \( \tau_x^e > 0 \) along the path of the ACC, this drives, via Eq.(10), surface eddy-induced circulation toward the pole, as sketched in the schematic diagram, Fig.10. In contrast, the absolute value of \( \tau_x^w \) decreases in magnitude with depth. Along the path of the ACC \( \tau_x^w > 0 \), but now an equatorward Ekman flow is driven, also sketched in Fig.10. We thus again clearly see the apposing nature of the wind and eddy-forced ageostrophic residual flow in the surface layers of the ACC.

5 Conclusions and discussion

We have inferred effective diffusivities by numerically monitoring the lengthening of tracer contours — Eq.(4) — as they are strained by surface geostrophic flow in the southern ocean observed by satellite altimetry. The theoretical background to the calculation is that due to Nakamura (1996). It is reviewed and connections made to the Walin (1982) formulation of water mass transformation in an Appendix.

The following broad conclusions can be drawn which, we feel, will stand the test of time and transcend the particular methods and implementation details used in our study:

1. effective diffusivities associated with the lateral stirring of properties at the sea surface due to interior baroclinic instability, show considerable spatial variability — by a factor of at least 2 — and are large on the equatorward flank of the ACC and are small at the jet axis — see Fig.4. This signature is a very robust result and is not sensitive to numerical details.
2. we expect, and indeed observe, that regions of high and low effective diffusivity are collocated with regions of, respectively, weak and strong isentropic PV gradients. This is a common feature of geophysical flows. Parcels of fluid can more readily disperse, and hence mix their properties, where PV gradients are weak rather than large.

3. eddy stresses drive surface residual flow toward the pole and wind stresses drive residual surface fluid away from the pole, as sketched in Fig.10. Surface convergence and subsequent subduction can thus be expected to take place in the vicinity of the axis of the ACC, the exact location depending on the relative magnitude of the eddy stresses and wind stresses.

4. in the 10 to 250 $P_e$ number range employed in the present study — see Fig.3 — absolute magnitudes of effective diffusivity are found to be somewhat dependent on the magnitude of the microdiffusivity used to calculate them using Eq.(4). This dependency can be interpreted in the context of a real physical process — the smallscale diffusivity, $k$, associated with baroclinic instability local to the mixed layer acting on surface buoyancy gradients created by straining of surface properties by the interior geostrophic eddy field. Depending on the magnitude of the $k$ assumed, we find that effective diffusivities range in magnitude across the ACC from $2000 \, \text{m}^2\,\text{s}^{-1}$ and $500 \, \text{m}^2\,\text{s}^{-1}$ — see Fig.4.

5. for the aforementioned range of effective diffusivities, eddy stresses are comparable in magnitude to wind-stresses — see Fig.9 — suggesting that eddy processes play a zero-order role in the streamwise averaged dynamics of the ACC.

6 **Acknowledgements**

The altimeter products were produced by the CLS Space Oceanography Division in France. We thank our colleagues at MIT and Cambridge University — Raf Ferrari, Alan Plumb and Peter Haynes — for many useful conversations and comments, and also Bernard Legras of the Ecole Normale Superieure.
Figure 10: The ACC flows along a front in the large-scale IPV field. Eddy stresses drive surface residual flow poleward toward the axis of the ACC and wind stresses drive residual surface fluid equatorward. Surface convergence and subsequent subduction can thus be expected to take place in the vicinity of the axis of the ACC, the exact location depending on the relative magnitude of the eddy stresses and wind stresses.
7 Appendix

7.1 Topex data

The time series of altimetric observations are Sea Level Anomaly (SLA) maps obtained from the final combined processing of Topex/Poseidon and ERS-1/2 data. There is one map every 10 days for a period of 5 years (October 1992 to October 1997). Anomaly maps were obtained using an improved space/time objective analysis method which takes into account long wavelength errors correlated noise. The method is described in Le Traon, Nadal and Ducet (1998). For each grid point, data inside a time window of ±10 days for T/P and ±18 days for ERS, and a space window of ±3 ZC, are used. The maps have a resolution of 0.25 degree by 0.25 degree.

Sea Level Anomalies are relative to a 3-year mean (January 1993 to January 1996). A specific processing is performed to obtain an ERS-1/2 mean consistent with T/P mean. Topex/Poseidon M-GDR (version C) recently reprocessed by AVISO were used. This version includes, in particular, the new JGM-3 orbits, the CSR3.0 tidal model and the correction of TOPEX drift. ERS-1/2 are the OPRs distributed by CERSAT. Altimetric corrections were updated to be homogenous with T/P and a global adjustment using T/P as a reference was performed to correct for ERS-1/2 orbit error. Additional information can be found in Le Traon and Ogor (1998).

The height relative to the geoid is obtained by subtracting the geoid model described in Lemoine et al (1997).

7.2 Derivation of $K_{eff}$ formulae

To make contact points between the meteorological and oceanographic literature we review and derive Nakamura’s expression, Eq.(3), beginning from the water mass transformation formalism due to Walin (1982). Our starting point is the tracer advection equation, Eq.(2), which we write in the form

$$\frac{\partial q}{\partial t} = -\nabla \cdot (N_q + qv)$$

where $N_q$ is the non-advective (diffusive) flux of $q$:

$$N_q = -k\nabla q$$  \hspace{1cm} (14)

The volume flux across $q$ contours defined by
\[ a = \oint (\mathbf{v} - \mathbf{v}_q) \cdot \mathbf{n}_q \, dl \]

where \( \mathbf{n}_q = \frac{\nabla q}{|\nabla q|} \) is a unit vector normal to \( q \) contours and \( \mathbf{v}_q = -\frac{n_q}{|\nabla q|} \frac{\partial q}{\partial t} \) is the velocity of a \( q \) contour normal to itself, can only be associated with non-advective fluxes (because the advecting velocity field is non-divergent) — see Fig.11. This volume flux can be related to diffusive fluxes thus (see Garrett and Tandon, 1995; Marshall et al, 1999, Eq.3.7)

\[ a = \frac{\partial A}{\partial t} = -\frac{\partial D}{\partial q} \quad (15) \]

where \( D \) is the diffusive flux across a \( q \) contour given by\(^4\):

\[ D = \oint \mathbf{N}_q \cdot \mathbf{n}_q \, dl = \oint \mathbf{N}_q \cdot \frac{\nabla q}{|\nabla q|} \, dl = \frac{\partial}{\partial q} \int \mathbf{N}_q \cdot \nabla q \, dA \]

For an \( \mathbf{N}_q \) given by Eq.(14) then

\[ D = -k \frac{\partial}{\partial q} \int |\nabla q|^2 \, dA \]

and Eq.(15) becomes

\[ \frac{\partial A}{\partial t} = -\frac{\partial D}{\partial q} = k \frac{\partial}{\partial q} \frac{\partial}{\partial A} \int |\nabla q|^2 \, dA \]

Now, since \( A = A(q) \), then \( \frac{\partial A}{\partial t} = \frac{\partial A}{\partial q} \frac{\partial q}{\partial t} \); \( \frac{\partial}{\partial q} = \frac{\partial A}{\partial q} \frac{\partial}{\partial A} \) and the above may be written as a diffusion equation in area coordinates thus:

\[ \frac{\partial q}{\partial t} = \frac{\partial}{\partial A} \left( \kappa_{\text{eff}}(A) \frac{\partial q}{\partial A} \right) \quad (16) \]

with diffusion coefficient

\[ \kappa_{\text{eff}}(A) = k \frac{1}{\left( \frac{\partial q}{\partial A} \right)^2} \frac{\partial}{\partial A} \int |\nabla q|^2 \, dA. \quad (17) \]

\(^4\)The following identity (a generalized form of Leibnitz theorem) is used

\[ \frac{\partial}{\partial q} \int_{\mathcal{A}} c(x, t) \, dA = \oint_{L} c(x, t) \frac{1}{|\nabla q|} \, dl \]
Thus $q$ diffuses in $A$ coordinates at rate $\kappa_{eff}$ which, according to Eq.(17), can be computed by summing up $|\nabla q|^2$ between adjacent $q$ contours. As we now discuss, it can also be related to the length of $q$ contours strained by the velocity field.

Note that $\kappa_{eff}$ has units of $m^4s^{-1}$ i.e. that of diffusivity $\times$ length$^2$: the diffusivity in area coordinates is — see Eq.(17) — $kL_{eq}^2$ where $L_{eq}$, known as the ‘equivalent length’, is given by:

$$L_{eq}^2 = \left(\frac{\partial A}{\partial q}\right)^2 \frac{\partial}{\partial A} \int |\nabla q|^2 dA = \frac{\partial A}{\partial q} \frac{\partial}{\partial q} \int |\nabla q|^2 \frac{dl dq}{|\nabla q|} = \left[\frac{\partial}{\partial A} \int |\nabla q| dA\right] dldq \left|\nabla q\right|$$

(18)

where we have used the relation $\frac{\partial A}{\partial q} = \frac{\partial}{\partial q} \int dA = \frac{\partial}{\partial q} \int \frac{dl dq}{|\nabla q|} = \int \frac{dl}{|\nabla q|}$.

Since $L = \int dl$, then $L_{eq}^2 = L^2$, the actual length of a $q$ contour, if $|\nabla q| = \text{const on } dl$. If $|\nabla q| \neq \text{const on } dl$ then $L_{eq}^2 \geq L^2$, as discussed in Haynes and Shuckburgh (2000). This can be seen as follows. Writing $L = \int |\nabla q| \frac{dl}{|\nabla q|} = \frac{\partial}{\partial q} \int |\nabla q| dA = \frac{1}{\left(\frac{\partial A}{\partial A}\right)} \frac{\partial}{\partial A} \int |\nabla q| dA$ we see that:

$$L_{eq}^2 = \frac{1}{\left(\frac{\partial A}{\partial A}\right)^2} \frac{\partial}{\partial A} \int |\nabla q|^2 dA \geq L^2 = \left(\frac{1}{\left(\frac{\partial A}{\partial A}\right)} \frac{\partial}{\partial A} \int |\nabla q| dA\right)^2$$

because the ‘sum of the squares’ is always greater than ‘the square of the sum’.

To obtain a diffusivity with conventional units we define a quantity

$$K_{eff} = \frac{kL_{eq}^2}{L_{min}^2}$$

(19)

where $L_{min}$ is the minimum length of a $q$ contour. Since $L_{eq}^2 \geq L^2 \geq L_{min}^2$, geometrically $\kappa_{eff}$ may be interpreted as the enhancement of diffusion due to the enhanced complexity of the tracer contours. On a sphere, the minimum length of a tracer contour is given by $L_{min} = 2\pi r \cos \phi_e$, where $\phi_e$ is known as the ‘equivalent latitude’, this being the slowest decaying mode of the
diffusion equation on a sphere. For application to the Southern Ocean, the continental boundaries prevent this minimum state being reached and the slowest decaying mode of the diffusion equation is not given by a simple analytic formula. We therefore obtain $L_{\text{min}}$ numerically.

### 7.3 Residual-mean theory

The residual momentum equation can be written in the planetary geostrophic limit (see, Ferreira, Marshall and Heimbach, 2005) thus:

$$f \hat{z} \times \mathbf{v}_{\text{res}} = \frac{1}{\rho_{\text{ref}}} \nabla p + \frac{1}{\rho_{\text{ref}}} \frac{\partial (\tau^w + \tau^e)}{\partial z}$$

(20)

where $\mathbf{v}_{\text{res}}$ is the residual flow, the sum of mean and eddy contributions,

$$\mathbf{v}_{\text{res}} = \overline{\mathbf{v}} + \mathbf{v}^*$$

(21)

and the eddy-induced velocity $\mathbf{v}^*$ can be expressed in terms of a vector streamfunction thus,

$$\mathbf{v}^* = -\nabla \times \Psi$$

(22)

defined by

$$\Psi = (\Psi_x, \Psi_y) = \left( \frac{\overline{u'v'}}{b_z}, -\frac{\overline{u'v'}}{b_z}, 0 \right).$$

(23)
The eddy stress is related to the eddy streamfunction thus:

\[ \tau^e = (\tau^e_x, \tau^e_y) = \rho_{ref} f \Psi \]  

(24)

where \( \Psi \) is given by Eq.(23).

The associated residual-mean buoyancy equation is:

\[ \frac{\partial \bar{b}}{\partial t} + \mathbf{v}_{res} \cdot \nabla \bar{b} = -\nabla \cdot \mathbf{F}_{res} + \mathbf{S} \]  

(25)

where \( \mathbf{S} \) are diabatic sources and sinks and \( \mathbf{F}_{res} = \frac{\rho \mathbf{v}_{0} \mathbf{b}_{0}}{b_z} \mathbf{z} \) are residual eddy fluxes with \( \mathbf{z} \) a unit vector in the vertical.

Complications arise near boundaries. Following Treguier et al (1997), the domain is divided into an adiabatic interior where the eddy flux is skew (\( \mathbf{F}_{res} \longrightarrow 0 \)) and a surface layer of depth \( h_s \) where eddy fluxes develop a diabatic component as they become parallel to the boundary and isopycnal surfaces steepen under the influence of turbulent mixing and air-se fluxes. In this surface layer the definition of \( \Psi \) is modified by assuming that the return flow is spread over the diabatic layer:

\[ \Psi_{z=-h_s} = \left( \frac{\rho \mathbf{v}_{0}}{b_z}, \frac{-\rho \mathbf{v}_{0}}{b_z}, 0 \right)_{z=-h_s} \mu, \quad -h_s < z < 0 \]  

(26)

where \( \mu \) changes from 0 at the surface to 1 at \( z = -h_s \). This ensures that \( \Psi = 0 \) and hence that \( \tau^e = 0 \) on the boundary.

8 References


32


