Chapter 13

Appendices

13.1 Derivations

13.1.1 The Planck function

A black body is a theoretical construct that absorbs 100% of the radiation that hits it. Therefore it reflects no radiation and appears perfectly black. It is also a perfect emitter of radiation. Planck showed that the power per unit area, per unit solid angle, per unit wavelength, emitted by a black body is given by:

\[ B_\lambda(T) = \frac{2hc^2}{\lambda^5 \left( \exp \left[ \frac{hc}{\lambda kT} \right] - 1 \right)} \]

where \( h \) is Planck’s constant, \( c \) is the speed of light, \( k \) is Boltzmann’s constant and \( \lambda \) is the wavelength of the radiation.

Figs.2.2 and 2.3 are plots of \( B_\lambda(T) \) against wavelength for various \( T \)'s.

If \( B_\lambda \) is integrated over all wavelengths, one obtains the black-body radiance:

\[ \int_0^\infty B_\lambda(T) d\lambda = \frac{\sigma}{\pi} T^4 \]

where \( \sigma = \frac{2\pi^5 k^4}{15hc^2} \) is the Stefan-Boltzmann constant. The above can be written as an integral over \( \ln \lambda \) thus:
\[ T^{-4} \int_0^\infty \lambda B_\lambda(T) d\ln \lambda = \frac{\sigma}{\pi} \]

So if \( T^{-4} \lambda B_\lambda \) is plotted against \( \ln \lambda \) then the area under the curve is independent of \( T \) — this is the form plotted in Fig. 2.5. For more details see Andrews (2000).

### 13.1.2 Computation of available potential energy

Consider the two-layer fluid shown in Fig. 8.9 in which the interface is given by:

\[ h = \frac{1}{2} H + \gamma y . \]

The potential energy of the system is (noting that \( \gamma < \frac{H}{2} \) i.e. the interface does not intersect the upper or lower boundaries)

\[
P = \int_0^H \int_{-L}^L g\rho z \, dy \, dz
= g \int_{-L}^L dy \left[ \int_0^{h(y)} \rho z \, dz + \int_{h(y)}^H \rho_1 z \, dz \right]
= g\rho_1 \int_{-L}^L dy \int_0^H z \, dz + g \Delta \rho \int_{-L}^L dy \int_0^{h(y)} z \, dz
= gH^2 L \rho_1 + g \frac{\Delta \rho}{2} \int_{-L}^L h^2(y) \, dy .
\]

Substituting for \( h \) we have

\[
\int_{-L}^L h^2(y) \, dy = \int_{-L}^L \left[ \frac{1}{4} H^2 + \lambda y + \lambda^2 y^2 \right] \, dy
= \frac{1}{2} H^2 L + \frac{2}{3} \lambda^2 L^3 ,
\]

and so

\[
P = gH^2 L \left( \rho_1 + \frac{\Delta \rho}{4} \right) + g \frac{\Delta \rho}{3} \gamma^2 L^3
\]

which, when expressed in terms of reduced gravity, \( g' = g \frac{(\rho_1 - \rho_2)}{\rho_1} \), is Eq. 8.9.
13.1.3 Internal energy for a compressible atmosphere

Internal energy for a perfect gas is defined by,

\[ IE = c_v \int \rho T \, dV = \frac{c_v}{R} \int p \, dV = \frac{c_v}{R} \int dA \int_{zs}^\infty p \, dz, \]

where \( dA \) is the area element such that \( dV = dA \, dz \), and where \( zs \) is the height of the Earth’s surface. If we neglect surface topography, so that \( zs = 0 \), then, integrating by parts,

\[ \int_0^\infty p \, dz = \left[ zp \right]_0^\infty - \int_0^\infty \frac{\partial p}{\partial z} \, z \, dz. \]

Since we saw in Chapter 3 that pressure decays approximately exponentially with height, \( (zp) \to 0 \) as \( z \to \infty \). Therefore, using hydrostatic balance, Eq.(3.3), we have

\[ \int_0^\infty p \, dz = g \int_0^\infty \rho z \, dz, \]

and so the internal energy is

\[ IE = \frac{c_v}{R} g \int z \rho \, dV. \]

13.2 Laboratory experiments

13.2.1 The Complete List

GFD 0: Rigidity imparted to rotating fluids — Section 0.2.2.
GFD I: Cloud formation on adiabatic expansion — Section 1.3.2.
GFD II: Convection — Section 4.2.4.
GFD III: Radial inflow — Section 6.6.1.
GFD IV: Studies of parabolic equipotential surfaces — Section 6.6.4.
GFD V: Inertial Circles — Section 6.6.4.
GFD VI: Perrot’s bathtub experiment — Section 6.6.5.
GFD VII: Taylor Columns — Section 7.2.1.
GFD VIII: Thermal Wind and Hadley Circulation — Section 7.3.1.
GFD IX: Cylinder ‘collapse’ under gravity and rotation — Section 7.3.3.
GFD X: Ekman layers — Section 7.4.1.
GFD XI: Baroclinic instability in a dishpan — Section 8.2.2.
### 13.2.2 Measurement of table rotation rates

The rotation rate of our table can be quoted in various units. The following are often used — period, revolutions per minute, or units of ‘f’, as described below and set out in Table 13.1:

1. The angular velocity of the tank, $\Omega$, in radians per second
2. The Coriolis parameter ($f$) defined as $f = 2\Omega$
3. The period of one revolution of the tank is $\tau = \frac{2\pi}{\Omega}$
4. Revolutions per minute, rpm = $\frac{60}{\tau}$

Note that if $\Omega$ is the rate of rotation of the tank in radians per second, then the period of rotation is $\tau_{\text{tank}} = \frac{2\pi}{\Omega}$ s. Thus if $\Omega = 1$, $\tau_{\text{tank}} = 2\pi$ s.

<table>
<thead>
<tr>
<th>$\Omega$ (rad/s)</th>
<th>0</th>
<th>0.25</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f = 2\Omega$ (rad/s)</td>
<td>0</td>
<td>0.5</td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>$\tau$ (s)</td>
<td>$\infty$</td>
<td>8$\pi$</td>
<td>$4\pi$</td>
<td>$2\pi$</td>
<td>$\frac{4\pi}{3}$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>rpm</td>
<td>0</td>
<td>2.4</td>
<td>4.7</td>
<td>9.5</td>
<td>14.3</td>
<td>19.1</td>
</tr>
</tbody>
</table>

Table 13.1: Various measures of spin rates.

GFD XII: Ekman pumping and suction — Section 10.1.2.
GFD XIII: Wind-driven ocean gyres — Section 10.2.3.
GFD XIV: Thermohaline Circulation — Section 11.3.2.

### 13.3 Mathematical definitions and notation

#### 13.3.1 Vector identities

Cartesian coordinates are best used for getting our ideas straight, but occasionally we also make use of polar (sometimes called cylindrical) and spherical
polar coordinates (see below).

The Cartesian coordinate system is defined by three axes at right angles to each other. The horizontal axes are labeled \(x\) and \(y\), and the vertical axis is labeled \(z\), with associated unit vectors \(\hat{x}, \hat{y}, \hat{z}\) (respectively). To specify a particular point we specify the \(x\) coordinate first (abscissa), followed by the \(y\) coordinate (ordinate), followed by the \(z\) coordinate, to form an ordered triplet \((x, y, z)\).

If \(\phi\) is a vector field and \(\mathbf{a}\) a 3-dimensional vector field thus:

\[
\mathbf{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}
\]

where \(a_x, a_y\) and \(a_z\) are magnitudes of the projections of \(\mathbf{a}\) along the three axes, then:

I. \(\nabla \phi = \hat{x} \frac{\partial \phi}{\partial x} + \hat{y} \frac{\partial \phi}{\partial y} + \hat{z} \frac{\partial \phi}{\partial z}\) [the "gradient of \(\phi\)" or "grad \(\phi\)", a vector]

II. \(\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\) [the "divergence of \(\mathbf{a}\)" or "div \(\mathbf{a}\)", a scalar]

III. \(\nabla \times \mathbf{a} = \hat{x} \left( \frac{\partial a_y}{\partial z} - \frac{\partial a_z}{\partial y} \right) + \hat{y} \left( \frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z} \right) + \hat{z} \left( \frac{\partial a_x}{\partial y} - \frac{\partial a_y}{\partial x} \right)\) [the "curl of \(\mathbf{a}\)", a vector]

Inspection of I, II, and III shows that we can define the operator \(\nabla\) uniquely as:

IV. \(\nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}\)

The quantity \(\nabla \cdot (\nabla \phi)\) is:

\[
\nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi
\]

V. \(\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z\) [the scalar product of \(\mathbf{a}\) and \(\mathbf{b}\), a scalar]

VI \(\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{x}(a_y b_z - a_z b_y) + \hat{y}(a_z b_x - a_x b_z) + \hat{z}(a_x b_y - a_y b_x)\) [the vector product of \(\mathbf{a}\) and \(\mathbf{b}\), a vector].

Here \(\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial \sigma}{\partial x} & \frac{\partial \sigma}{\partial y} & \frac{\partial \sigma}{\partial z} \end{vmatrix}\) is the determinant. Thus, for example, in Eq.(7.16),

\[
\hat{z} \times \nabla \sigma = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ \frac{\partial \sigma}{\partial x} & \frac{\partial \sigma}{\partial y} & \frac{\partial \sigma}{\partial z} \end{vmatrix} = \hat{x} \left( -\frac{\partial \sigma}{\partial y} \right) + \hat{y} \left( \frac{\partial \sigma}{\partial x} \right).
\]

Some useful identities are:
1. $\nabla \cdot (\nabla \times \mathbf{a}) = 0$
2. $\nabla \times (\nabla \phi) = 0$
3. $\nabla \cdot (\phi \mathbf{a}) = \phi \nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla \phi$
4. $\nabla \times (\phi \mathbf{a}) = \phi \nabla \times \mathbf{a} + \nabla \phi \times \mathbf{a}$
5. $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$
6. $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\nabla \cdot \mathbf{b}) - \mathbf{b} (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla) \mathbf{a} - (\mathbf{a} \cdot \nabla) \mathbf{b}$
7. $\nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$
8. $\nabla \times (\nabla \times \mathbf{a}) = \nabla (\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$

In (8), $\nabla^2 \mathbf{a} = \hat{x} \nabla^2 a_x + \hat{y} \nabla^2 a_y + \hat{z} \nabla^2 a_z$.

An important special case of (7) arises when $\mathbf{a} = \mathbf{b}$:

$$(\mathbf{a} \cdot \nabla) \mathbf{a} = \nabla \left( \frac{1}{2} \mathbf{a} \cdot \mathbf{a} \right) - \mathbf{a} \times (\nabla \cdot \mathbf{a}).$$

These relations can be verified by the arduous procedure of applying the definitions of $\nabla$, $\nabla \cdot$, $\nabla \times$ (and $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ as necessary) to all the terms involved.

### 13.3.2 Other orthogonal coordinate systems

#### Polar coordinates

Any point is specified by the distance $r$ (radius vector pointing outwards) from the origin and $\theta$ (vectorial angle) measured from a reference line ($\theta$ is positive if measured counterclockwise, and negative if measured clockwise) — see Fig.6.7.

$$x = r \cos \theta; \quad y = r \sin \theta$$

The velocity vector is $\mathbf{u} = \hat{r} v_r + \hat{\theta} v_\theta$ where $v_r = \frac{Dr}{Dt}$ and $v_\theta = r \frac{D\theta}{Dt}$ are the radial and azimuthal velocities, respectively.

In polar coordinates:

$$\nabla \phi = \hat{r} \frac{\partial \phi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \phi}{\partial \theta}$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$